Charged Particle Tracking Through Magnetic Fields
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This note describes a method of determining the trajectory of a charged particle in an arbitrary magnetic field where the problem can be parameterized in terms of changes to one of the orthonormal Cartesian coordinates. This is the case in fixed target experiments with high-ish momentum tracks traversing analyzing magnet fields perpendicular to the beam axis. The method described below requires the field in only two places along the integration step instead of the four required for fourth order Runge-Kutte integration, a modified form of which is used in GEANT. This is true because the exact second derivative of the motion can be expressed in the trajectory parameters. This expression is then integrated twice using a simple Euler integration scheme.

Equations of Motion

The Lorentz equation gives the classical equations of motion for a charged particle in a magnetic field:

$$F = \frac{q}{c}(v \times B)$$

which can be re-written in differential form as:

$$dp = \frac{q}{c}\left(d\ell \times B\right)$$

that are the set of equations which must be integrated to obtain the track trajectory. We also use the following definitions of the trajectory:

$$\begin{aligned}
p^2 &= p_z^2\left(1 + x'^2 + y'^2\right) \\
x' &= p_x / p_z \\
y' &= p_y / p_z
\end{aligned}$$

and the fact that the change of momentum magnitude is zero for this motion (a static magnetic field cannot do work).

The three component equations of motion are then:

$$dp_x = \frac{q}{c}\left(B_z dy - B_y dz\right)$$

$$dp_y = \frac{q}{c}\left(B_x dz - B_z dx\right)$$

$$dp_z = \frac{q}{c}\left(B_y dx - B_x dy\right)$$

this particular method re-expresses these equations in terms of changes along the z-axis:

$$dp_x = \frac{q}{c} dz \left(B_z \frac{dy}{dz} - B_y \right)$$

$$dp_y = \frac{q}{c} dz \left(B_x - B_z \frac{dx}{dz} \right)$$

$$dp_z = \frac{q}{c} dz \left(B_y \frac{dx}{dz} - B_x \frac{dy}{dz} \right)$$

upon substitution this becomes:

$$dp_x = \frac{q}{c} dz \left(B_z \frac{p_y}{p_z} - B_y \right)$$

$$dp_y = \frac{q}{c} dz \left(B_x - B_z \frac{p_x}{p_z} \right)$$

$$dp_z = \frac{q}{c} dz \left(B_y \frac{p_x}{p_z} - B_x \frac{p_y}{p_z} \right)$$

two additional relations of use are:

$$x'' = \frac{d}{dz} \frac{p_x}{p_z} = \frac{1}{p_z} \frac{dp_x}{dz} - \frac{p_x}{p_z^2} \frac{dp_z}{dz}$$

$$y'' = \frac{d}{dz} \frac{p_y}{p_z} = \frac{1}{p_z} \frac{dp_y}{dz} - \frac{p_y}{p_z^2} \frac{dp_z}{dz}$$

upon substitution of the equations of motion they become:

$$x'' = \frac{q/c}{p} \frac{1}{\sqrt{1 + x'^2 + y'^2}} \left[y' B_z + \left(1 - x'^2\right) B_y + x' y' B_x \right]$$

$$y'' = \frac{q/c}{p} \frac{1}{\sqrt{1 + x'^2 + y'^2}} \left[-x' B_z + \left(1 + y'^2\right) B_x - x' y' B_y \right]$$

which gives the exact second derivative of the trajectory given the momentum magnitude and "cosines", and the magnetic field at any point. The parameters that describe the track are then:

$$\text{parameters} = \left(\frac{q/c}{p}, x_0, y_0, x'_0, y'_0 \right).$$

To integrate to some point, start at a known point with a known momentum (translate to slope) and step along the z-direction **Δz**. The notation used here is that a step in **Δz** corresponds to incrementing a subscript from *i* to *i+1* with the "obvious" convention that a half-increment step, **Δz**, is represented by the subscript *i+1/2*. The further simplification of notation involves the field variables:

$$B_i \equiv B(x_i, y_i, z_i)$$

$$x_i \equiv x(z_i), \quad y_i \equiv y(z_i)$$

The trajectory integration proceeds by stepping halfway, correcting the slopes for the intermediate value of the magnetic field, and then stepping the remaining distance. The position at halfway is given by:

$$x_{i+1/2} = x_i + 1/2\,\Delta z\, x'_i$$

$$y_{i+1/2} = y_i + 1/2\,\Delta z\, y'_i$$

these values are then used to calculate the new values of the field which, in turn, are used to calculate the new value of the acceleration. The slope values are updated as:

$$x'_{i+1/2} = x'_i + 1/2\,\Delta z\, x''_{i+1/2}$$

$$y'_{i+1/2} = y'_i + 1/2\,\Delta z\, y''_{i+1/2}$$

then used to step the rest of the way:

$$x_{i+1} = x_{i+1/2} + 1/2\,\Delta z\, x'_{i+1/2}$$

$$y_{i+1} = y_{i+1/2} + 1/2\,\Delta z\, y'_{i+1/2}$$

and finally:

$$x'_{i+1} = x'_{i+1/2} + 1/2\,\Delta z\, x''_{i+1}$$

$$y'_{i+1} = y'_{i+1/2} + 1/2\,\Delta z\, y''_{i+1}$$

which continues for each step.

The major advantage of these scheme is that only two magnetic field values are required per step. Obtaining the magnetic field can often involve a lot of time (interpolations, field calculations, etc.).